Exact Travelling Wave Solutions of Toda Lattice Equations Obtained via the Exp-Function Method

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We generalize the exp-function method, which was used to find new exact travelling wave solutions of nonlinear partial differential equations or coupled nonlinear partial differential equations, to nonlinear differential-difference equations. As illustration, we study two Toda lattices and obtain some new travelling wave solutions by means of the exp-function method. As some special examples, some new exact travelling wave solutions can degenerate into the kink-type solitary wave solutions reported in open literatures.

Key words: Toda Lattices; Exact Travelling Wave Solution; Exp-Function Approach.
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1. Introduction

Seeking for exact solutions of nonlinear evolution equations has flourished into a research area of great importance and interest during the last two decades. The investigation of exact solutions of nonlinear differential-difference equations (NDDEs) plays an important role in the study of nonlinear physical phenomena, and gradually becomes one of the most important and significant tasks. Recently, there has been some interest in exactly solvable discretized nonlinear problems, and the inverse scattering method has been extended to a wide class of such problems [1]. More recently, with the development of symbolic computation, many direct and effective methods are available to solve NDDEs. For instance, Baldwin et al. [2] derived the kink-type solutions of many spatially discrete nonlinear models in terms of the tanh-function. Dai et al. [3] obtained the kink-type solutions of the discrete sine-Gordon equation by means of the hyperbolic function approach. Some authors [4,5] extended the tanh-function approach to solve NDDEs. Yan [6] developed the discrete sine-Gordon expansion method in NDDEs. Wang and Zhang [7] used the bi-function method to solve the extended Lotka-Volterra equation and the discrete Korteweg-de Vries (KdV) equation. Moreover, the Jacobian elliptic function method was generalized to solve differential-difference equations [8 – 11].

In a recent study to solve nonlinear partial differential equations (NPDEs), the exp-function method [12, 13] was one of the most effective direct methods. How to extend these effective methods for NPDEs to solve NDDEs is an interesting and important issue. In the present paper, we study some Toda lattices to illustrate the exp-function method for NDDEs.

2. The Exp-Function Method for NDDEs

In this section, we would like to outline the exp-function method for NDDEs step by step.

Consider a system of M polynomial NDDEs

$$\begin{aligned}&\triangle(\mathbf{u}_{\mathbf{n}+\mathbf{p_1}}(\mathbf{x}),\ldots,\mathbf{u}_{\mathbf{n}+\mathbf{p_k}}(\mathbf{x}),\\&\mathbf{u}'_{\mathbf{n}+\mathbf{p_1}}(\mathbf{x}),\ldots,\mathbf{u}'_{\mathbf{n}+\mathbf{p_k}}(\mathbf{x}),\ldots,\\&\mathbf{u}^{(\mathbf{r})}_{\mathbf{n}+\mathbf{p_1}}(\mathbf{x}),\ldots,\mathbf{u}^{(\mathbf{r})}_{\mathbf{n}+\mathbf{p_k}}(\mathbf{x}))=0,\end{aligned} \tag{1}$$

where the dependent variable $\mathbf{u_n}$ has M components $u_{i,n}$, the continuous variable $\mathbf{x}$ has N components x_i, the discrete variable $\mathbf{n}$ has Q components n_j, the k shift vectors $\mathbf{p}_i \in Z^Q$, and $\mathbf{u}^{(\mathbf{r})}(\mathbf{x})$ denotes the collection of mixed derivative terms of order r.

According to the description of the tanh-function method in [2], the main steps of the exp-function method for NDDEs are outlined as follows.

Step 1: When we seek the travelling wave solutions of (1), the first step is to introduce the wave transformation $\mathbf{u}_{\mathbf{n}+\mathbf{p_s}}(\mathbf{x}) = \mathbf{U}_{\mathbf{n}+\mathbf{p_s}}(\xi_\mathbf{n})$, $\quad \xi_\mathbf{n} = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} c_j x_j + \zeta$ for any s $(s = 1,\cdots,k)$, where the coefficients $c_1, c_2, \cdots, c_N, d_1, d_2, \cdots, d_Q$ and the phase ζ

are all constants. In this way, (1) becomes

$$\begin{aligned}&\triangle(\mathbf{U_{n+p_1}}(\xi_\mathbf{n}),\ldots,\mathbf{U_{n+p_k}}(\xi_\mathbf{n}),\\&\mathbf{U'_{n+p_1}}(\xi_\mathbf{n}),\ldots,\mathbf{U'_{n+p_k}}(\xi_\mathbf{n}),\ldots,\\&\mathbf{U^{(r)}_{n+p_1}}(\xi_\mathbf{n}),\ldots,\mathbf{U^{(r)}_{n+p_k}}(\xi_\mathbf{n}))=0.\end{aligned} \tag{2}$$

Step 2: We propose the following series expansion as a solution of (2):

$$\mathbf{U_n}(\xi_\mathbf{n})=\frac{\sum_{l=-p}^{q}a_l\exp(l\xi_\mathbf{n})}{\sum_{m=-e}^{f}b_m\exp(m\xi_\mathbf{n})}, \tag{3}$$

where p,q,e and f are positive integers, which are given by the homogeneous balance principle, a_l and b_m are unknown constants to be determined. To determine the values of f and q, we balance the linear term of highest order in (2) with the highest-order nonlinear term. Similarly to determine the values of e and p, we balance the linear term of lowest order in (2) with the lowest-order nonlinear term.

Simple computation leads to the identity

$$\xi_{\mathbf{n}+\mathbf{p_s}}=\xi_\mathbf{n}+\varphi_s,$$

with

$$\varphi_s=p_{s1}d_1+p_{s2}d_2+\cdots+p_{sQ}d_Q, \tag{4}$$

where p_{sj} is the j-th component of shift vector $\mathbf{p}_s$.

Thus,

$$\mathbf{U_{n+p_s}}(\xi_\mathbf{n})=\frac{\sum_{l=-p}^{q}a_l\exp[l(\xi_\mathbf{n}+\varphi_s)]}{\sum_{m=-e}^{f}b_m\exp[m(\xi_\mathbf{n}+\varphi_s)]}. \tag{5}$$

Remark: Unlike difference equations which are fully discretized and differential equations which are fully continuous, NDDEs are semi-discretized with some (or all) of their spatial variables discretized while time is usually kept continuous. The key, how to apply familiar methods of the continuous case into the discrete case, is to search iterative relations between lattice indices, for example, the relations between indices n and $n+1$. That is to say, the key is seeking the iterative relation between (3) and (5) with (4).

Step 3: Determine the degrees p,q,e and f of the polynomial solutions (3) and (5). We can easily get the degrees p,q,e and f in ansatz (3) and ansatz (5) by balancing the highest nonlinear term and the highest-order derivative term in $\mathbf{U_n}(\xi_\mathbf{n})$ as in the continuous case.

Step 4: Substituting ansatz (3) and ansatz (5) into (2), then setting the coefficients of all independent terms in $\exp(j\xi_\mathbf{n})$ $(j=1,2,\ldots)$ to zero, we will get a system of algebraic equations. From the highly nonlinear and parameterized algebraic equations, the corresponding undetermined constants are explicitly determined by using Maple and Wu's elimination method [14].

Step 5: Substitute the values obtained in Step 4 into expression (3), and one can find the solutions of (1). To assure the correctness of the solutions, it is necessary to substitute them into the original equation (1).

3. Exact Travelling Wave Solutions of Some Toda Lattices

In this section, we apply the method developed in the preceding section to some Toda lattices.

3.1. Example 1

The Toda lattice [15] is of the form

$$\frac{\mathrm{d}^2u_n}{\mathrm{d}t^2}=\left(\frac{\mathrm{d}u_n}{\mathrm{d}t}+1\right)(u_{n-1}-2u_n+u_{n+1}). \tag{6}$$

In this case, $\mathbf{u}=u,\mathbf{x}=\{x_1,x_2\}=\{t,x\},\mathbf{n}=n_1=n$ and $\mathbf{p_1}=p_1=-1,\mathbf{p_2}=p_2=0,\mathbf{p_3}=p_3=1,d_1=d,c_1=c$.

Now we assume that (6) has the solutions as ansatz (3). The homogeneous balance procedure in Step 2 of Section 2 leads to the results $q=f$, $p=e$. We can freely choose the values of p and q, but the final solution does not strongly depend on the choice of values of p and q [12, 13]. For simplicity, we set $q=f=1$ and $p=e=1$. Thus we give the following formal travelling wave solutions of (6):

$$\begin{aligned}u_n&=\frac{a_{-1}\exp(-\xi_n)+a_0+a_1\exp(\xi_n)}{b_{-1}\exp(-\xi_n)+b_0+b_1\exp(\xi_n)},\\u_{n+1}&=\frac{a_{-1}\exp(-\xi_n-d)+a_0+a_1\exp(\xi_n+d)}{b_{-1}\exp(-\xi_n-d)+b_0+b_1\exp(\xi_n+d)},\\u_{n-1}&=\frac{a_{-1}\exp(-\xi_n+d)+a_0+a_1\exp(\xi_n-d)}{b_{-1}\exp(-\xi_n+d)+b_0+b_1\exp(\xi_n-d)},\end{aligned} \tag{7}$$

with

$$\xi_n=dn+ct+\zeta, \tag{8}$$

where $a_0,a_1,a_{-1},b_0,b_1,b_{-1},d$ and c are constants to be determined later. Inserting the expressions (7) and

(8) into (6), clearing the denominator and eliminating the coefficients of independent terms in $\exp(j\xi_n)$ $(j = 1,2,\ldots)$, yields a system of algebraic equations, from which we obtain many travelling wave solutions of the Toda lattice equation (6). Here we only partially list these solutions.

Case 1.

$$\begin{aligned} &a_0 = b_0 = 0,\\ &a_{-1} = \left[\frac{a_1}{b_1} \mp 2\sinh(d)\right] b_{-1},\\ &c = \pm\sinh(d), \end{aligned} \tag{9}$$

where a_1, b_1, b_{-1} and d are arbitrary constants. From (7)–(9), we can obtain the exact travelling wave solution of the Toda lattice equation (6), i. e.,

$$u_n = \frac{\left[\frac{a_1}{b_1} \mp 2\sinh(d)\right] b_{-1}\exp(-\xi_n) + a_1\exp(\xi_n)}{b_{-1}\exp(-\xi_n) + b_1\exp(\xi_n)}, \tag{10}$$

where $\xi_n = dn \pm \sinh(d)t + \zeta$.

As a special example, when $a_1 = A_0 \pm \sinh(d)$, $b_{-1} = b_1 = 1$, the solution (10) is

$$u_n = A_0 \pm \sinh(d)\tanh(\xi_n), \tag{11}$$

which turns out to be Baldwin's [2] and Dai's [16] kink-type solitary wave solution as expressed in (11).

As another special example, when $a_1 = A_0 \pm \sinh(d)$, $b_1 = 1, b_{-1} = -1$, the solution (10) is

$$u_n = A_0 \pm \sinh(d)\coth(\xi_n), \tag{12}$$

which turns out to be Dai's [16] travelling wave solution as expressed in (12).

Case 2.

$$\begin{aligned} &a_{-1} = b_{-1} = 0,\\ &a_1 = b_1\left[\frac{a_0}{b_0} \pm 2\sinh\left(\frac{d}{2}\right)\right],\\ &c = \pm 2\sinh\left(\frac{d}{2}\right), \end{aligned} \tag{13}$$

where a_0, b_0, b_1 and d are arbitrary constants. From (7), (8) and (13), we can obtain the exact travelling wave solution of the Toda lattice equation (6), i. e.,

$$u_n = \frac{a_0 + b_1\left[\frac{a_0}{b_0} \pm 2\sinh\left(\frac{d}{2}\right)\right]\exp(\xi_n)}{b_0 + b_1\exp(\xi_n)}, \tag{14}$$

where $\xi_n = dn \pm 2\sinh\left(\frac{d}{2}\right)t + \zeta$.

Case 3.

$$\begin{aligned} &a_1 = b_1 = 0,\\ &a_0 = b_0\left[\frac{a_{-1}}{b_{-1}} \pm 2\sinh\left(\frac{d}{2}\right)\right],\\ &c = \pm 2\sinh\left(\frac{d}{2}\right), \end{aligned} \tag{15}$$

where b_0, a_{-1}, b_{-1} and d are arbitrary constants. From (7), (8) and (15), we can obtain the exact travelling wave solution of the Toda lattice equation (6), i.e.,

$$u_n = \frac{a_{-1}\exp(-\xi_n) + b_0\left[\frac{a_{-1}}{b_{-1}} \pm 2\sinh\left(\frac{d}{2}\right)\right]}{b_{-1}\exp(-\xi_n) + b_0}, \tag{16}$$

where $\xi_n = dn \pm 2\sinh\left(\frac{d}{2}\right)t + \zeta$.

Case 4.

$$\begin{aligned} &b_1 = \frac{b_0^2}{4b_{-1}},\\ &a_0 = \pm\sinh\left(\frac{d}{2}\right)b_0 + \frac{a_{-1}b_0}{b_{-1}},\\ &a_1 = \pm\frac{b_0^2}{2b_{-1}}\sinh\left(\frac{d}{2}\right) + \frac{a_{-1}b_0^2}{4b_{-1}^2},\\ &c = \pm 2\sinh\left(\frac{d}{2}\right), \end{aligned} \tag{17}$$

where a_{-1}, b_0, b_{-1} and d are arbitrary constants. From (7), (8) and (17), we can obtain a new exact travelling wave solution of the Toda lattice equation (6), i. e.,

$$\begin{aligned} u_n = \Big\{&4a_{-1}b_{-1}^2\exp(-\xi_n) + 4a_{-1}b_0b_{-1}\\ &\pm 4b_0b_{-1}^2\sinh\left(\frac{d}{2}\right)\\ &+ \left[a_{-1}b_0^2 \pm 2b_0^2b_{-1}\sinh\left(\frac{d}{2}\right)\right]\exp(\xi_n)\Big\}\\ \cdot&\Big\{4b_{-1}^3\exp(-\xi_n) + 4b_0b_{-1}^2 + b_0^2b_{-1}\exp(\xi_n)\Big\}^{-1}, \end{aligned} \tag{18}$$

where $\xi_n = dn \pm 2\sinh\left(\frac{d}{2}\right)t + \zeta$.

It is well known that soliton solutions are interesting and physical relevant. Here we take the solution (10) as example to further analyze its properties by some figures. The wave velocity "c" of the solution (10) changes with wave number "d" plotted in Figure 1a. The solution (10) is a kink-type solitary wave shown in Fig. 1b by selecting appropriate parameters. As can be seen from Fig. 1b, with the growth of the

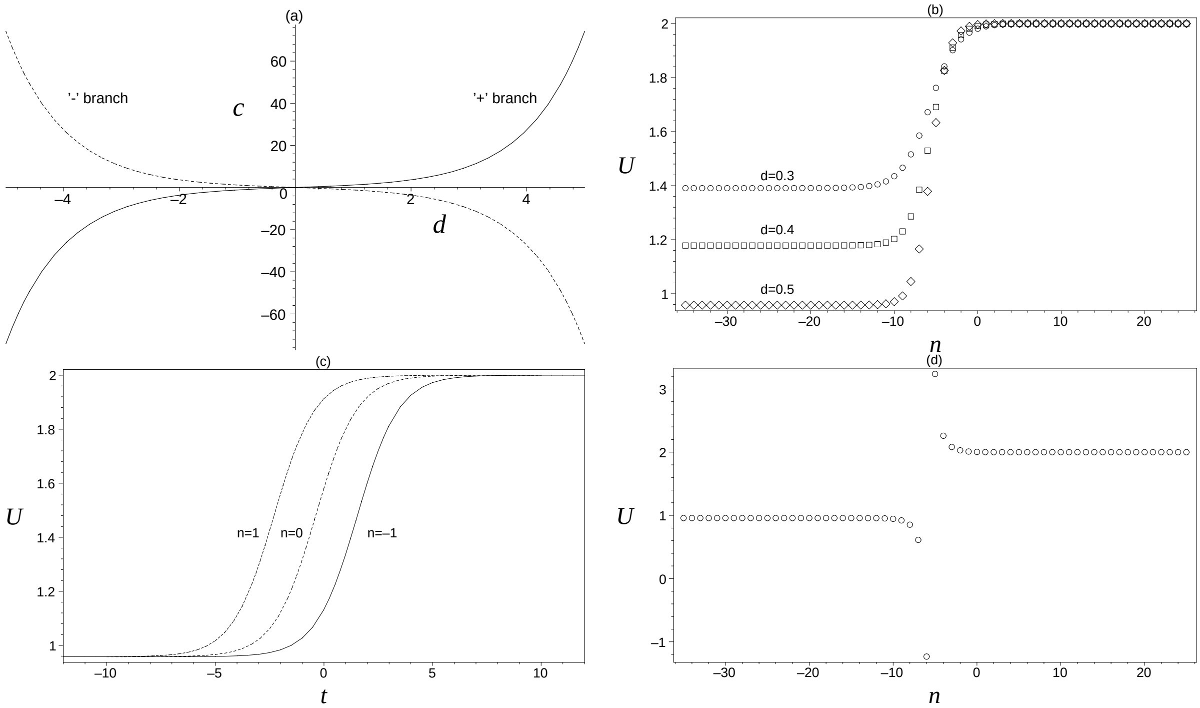

Fig. 1. (a) Wave velocity changes with wave number. (b) Kink-type solitary wave solution (10) with (+)-branch for the parameters $a_1 = 2$, $b_1 = b_{-1} = 1$, $\zeta = 0.2$ at time $t = 5$. (c) Asymptotic property of solution (10) with (+)-branch for the parameters $a_1 = 2$, $b_1 = b_{-1} = 1, d = 0.5$, $\zeta = 0.2$. (d) Exotic wave solution (10) with (+)-branch for the parameters $a_1 = 2$, $b_1 = 1$, $b_{-1} = -1$, $\zeta = 0.2$ at time $t = 5$.

wave number, the distance between the lattices becomes gradually bigger in the kink location of wave. Meanwhile, the asymptotic properties of the solution (10) with different lattice labels n are plotted in Figure 1c. Besides the property of the kink-type solitary wave, the solution (10) exhibits the feature of an exotic wave shown in Figure 1d. The mathematical expression of this exotic wave has the similar form as (12). However, different from the continuous case, the exotic wave doesn't exhibit singular properties although it is composed of the singular coth-function. When the parameters $b_1 b_{-1} > 0$ and $b_1 b_{-1} < 0$, the solution (10) is a kink-type solitary wave and an exotic wave, respectively.

3.2. Example 2

Another Toda lattice [17] has the form

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = u_n(v_n - v_{n-1}), \quad \frac{\mathrm{d}v_n}{\mathrm{d}t} = v_n(u_{n+1} - u_n). \quad (19)$$

In this case, $\mathbf{u} = \{u, v\}, \mathbf{x} = x_1 = t, \mathbf{n} = n_1 = n$ and $\mathbf{p_1} = p_1 = -1, \mathbf{p_2} = p_2 = 0, \mathbf{p_3} = p_3 = 1$.

The balance procedure admits us to give travelling wave solutions of (19) and, in case of $b_1 \neq 0, d_1 \neq 0$, this ansatz can be simplified as

$$\begin{aligned} u_n &= \frac{a_{-1}\exp(-\xi_n) + a_0 + a_1\exp(\xi_n)}{b_{-1}\exp(-\xi_n) + b_0 + \exp(\xi_n)}, \\ u_{n+1} &= \frac{a_{-1}\exp(-\xi_n - d) + a_0 + a_1\exp(\xi_n + d)}{b_{-1}\exp(-\xi_n - d) + b_0 + \exp(\xi_n + d)}, \quad (20) \\ u_{n-1} &= \frac{c_{-1}\exp(-\xi_n + d) + c_0 + c_1\exp(\xi_n - d)}{d_{-1}\exp(-\xi_n + d) + d_0 + \exp(\xi_n - d)}, \end{aligned}$$

$$\begin{aligned} v_n &= \frac{c_{-1}\exp(-\xi_n) + c_0 + c_1\exp(\xi_n)}{d_{-1}\exp(-\xi_n) + d_0 + \exp(\xi_n)}, \\ v_{n+1} &= \frac{c_{-1}\exp(-\xi_n - d) + c_0 + c_1\exp(\xi_n + d)}{d_{-1}\exp(-\xi_n - d) + d_0 + \exp(\xi_n + d)}, \quad (21) \\ v_{n-1} &= \frac{c_{-1}\exp(-\xi_n + d) + c_0 + c_1\exp(\xi_n - d)}{d_{-1}\exp(-\xi_n + d) + d_0 + \exp(\xi_n - d)}, \end{aligned}$$

with

$$\xi_n = dn + ct + \zeta. \quad (22)$$

Substituting ansatz (20) and ansatz (21) with (22) into (19), clearing the denominator and eliminating all the coefficients of the powers of $\exp(j\xi_n)$ $(j=1,2,\ldots)$ yields a nonlinear algebraic system. Proceeding as before we only partly list these solutions.

Case 1.

$$\begin{aligned}
&a_0=b_0=c_0=d_0=0,\\
&a_{-1}=\exp(2d)c_{-1},\quad a_1=-c\exp(-d)\operatorname{csch}(d),\\
&b_{-1}=d_{-1}=-\frac{c_{-1}}{c}\exp(d)\sinh(d),\\
&c_1=-c\exp(d)\operatorname{csch}(d),
\end{aligned}\tag{23}$$

where c_{-1}, d and c are arbitrary constants. From (20)–(23) we can obtain exact travelling wave solutions of the Toda lattice equation (19), i. e.,

$$\begin{aligned}
u_n&=\\
&\frac{\exp(2d)c_{-1}c\exp(-\xi_n)-c^2\exp(-d)\operatorname{csch}(d)\exp(\xi_n)}{-c_{-1}\exp(d)\sinh(d)\exp(-\xi_n)+c\exp(\xi_n)},\\
v_n&=\frac{c_{-1}c\exp(-\xi_n)-c^2\exp(d)\operatorname{csch}(d)\exp(\xi_n)}{-c_{-1}\exp(d)\sinh(d)\exp(-\xi_n)+c\exp(\xi_n)},
\end{aligned}\tag{24}$$

where $\xi_n=dn+ct+\zeta$.

Case 2.

$$\begin{aligned}
&a_0=b_0=c_0=d_0=0,\\
&a_{-1}=c_{-1},\\
&a_1=-c\exp(d)\operatorname{csch}(d),\\
&b_{-1}=-\frac{c_{-1}}{c}\exp(d)\sinh(d),\\
&d_{-1}=-\frac{c_{-1}}{c}\exp(-d)\sinh(d),\\
&c_1=-c\exp(-d)\operatorname{csch}(d),
\end{aligned}\tag{25}$$

where c_{-1}, d and c are arbitrary constants. From (20)–(22) and (25) we can obtain exact travelling wave solutions of the Toda lattice equation (19), i. e.,

$$\begin{aligned}
u_n&=\frac{c_{-1}c\exp(-\xi_n)-c^2\exp(d)\operatorname{csch}(d)\exp(\xi_n)}{-c_{-1}\exp(d)\sinh(d)\exp(-\xi_n)+c\exp(\xi_n)},\\
v_n&=\frac{c_{-1}c\exp(-\xi_n)-c^2\exp(-d)\operatorname{csch}(d)\exp(\xi_n)}{-c_{-1}\exp(-d)\sinh(d)\exp(-\xi_n)+c\exp(\xi_n)},
\end{aligned}\tag{26}$$

where $\xi_n=dn+ct+\zeta$.

Case 3.

$$\begin{aligned}
&a_{-1}=\frac{cd_0^2}{8}\exp(d)\operatorname{csch}(2d)\operatorname{csch}^2\left(\frac{d}{2}\right),\\
&b_{-1}=\frac{d_0^2}{4}\exp(d)\operatorname{csch}^2(d),\\
&c_{-1}=4cd_0^2\operatorname{csch}^3(d)\cosh(d),\\
&d_{-1}=\frac{d_0^2}{4}\operatorname{csch}^2(d),\quad c_0=0,\\
&a_0=-cd_0\exp\left(\frac{d}{2}\right)\operatorname{csch}\left(\frac{d}{2}\right)\left[1+\frac{1}{2}\operatorname{sech}(d)\right],\\
&b_0=\frac{d_0}{2}\exp\left(\frac{d}{2}\right)\operatorname{sech}\left(\frac{d}{2}\right),\\
&a_1=\frac{c}{2}\coth\left(\frac{d}{2}\right)\operatorname{sech}(d),\\
&c_1=-c\coth(d),
\end{aligned}\tag{27}$$

where d_0, c and d are arbitrary constants. From (20)–(22) and (27) we can obtain new exact travelling wave solutions of the Toda lattice equation (19), i. e.,

$$\begin{aligned}
u_n&=\Big\{cd_0^2\exp(d)\operatorname{csch}(2d)\operatorname{csch}^2\left(\frac{d}{2}\right)\exp(-\xi_n)\\
&\quad-cd_0\exp\left(\frac{d}{2}\right)\operatorname{csch}\left(\frac{d}{2}\right)\left[8+4\operatorname{sech}(d)\right]\\
&\quad+4c\coth\left(\frac{d}{2}\right)\operatorname{sech}(d)\exp(\xi_n)\Big\}\\
&\cdot\Big\{2d_0^2\exp(d)\operatorname{csch}^2(d)\exp(-\xi_n)\\
&\quad+4d_0\exp\left(\frac{d}{2}\right)\operatorname{sech}\left(\frac{d}{2}\right)+8\exp(\xi_n)\Big\}^{-1},\\
v_n&=\Big\{16cd_0^2\operatorname{csch}^3(d)\cosh(d)\exp(-\xi_n)\\
&\quad-4c\coth(d)\exp(\xi_n)\Big\}\\
&\cdot\Big\{d_0^2\operatorname{csch}^2(d)\exp(-\xi_n)+4d_0+4\exp(\xi_n)\Big\}^{-1},
\end{aligned}\tag{28}$$

where $\xi_n=dn+ct+\zeta$.

Figure 2a displays the solitary wave property of solution (24). From Fig. 2a, the physical quantity u is an kink-type solitary wave (squares), and v is an anti-kink solitary wave (circles). Similarly to solution (10), the solution (24) exhibits also the feature of an exotic wave as shown in Figure 2b. When $cdc_{-1}<0$ and $cdc_{-1}>0$, the solutions of (24) are kink-type solitary waves and exotic wave waves, respectively.

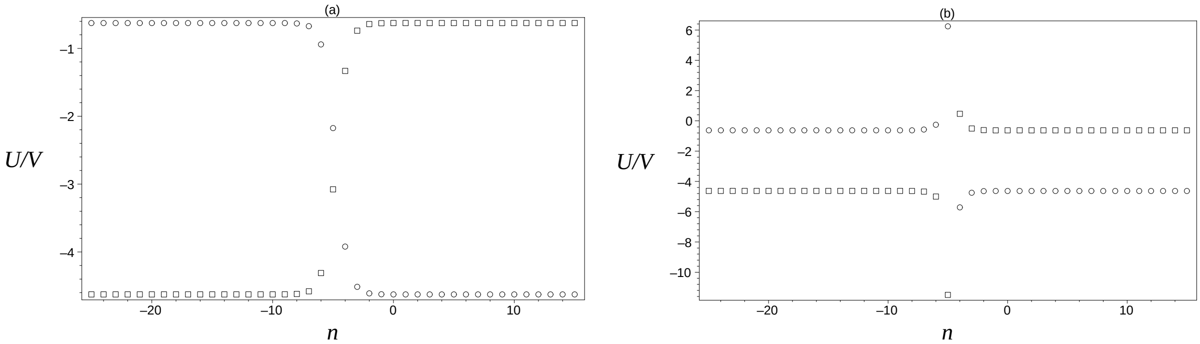

Fig. 2. (a) Kink-type solitary wave solution (24) for the parameters $c = 2, d = 1, c_{-1} = -0.2, \zeta = 0.2$ at time $t = 2$. (b) Exotic wave solution (24) for the parameters $c = 2, d = 1, c_{-1} = 0.2, \zeta = 0.2$ at time $t = 2$.

4. Summary and Discussion

In summary, we systematically illustrated the solution procedure of the exp-function method for nonlinear differential-difference equations. The solution procedure is very simple, and the obtained solution is very concise. We obtained some new exact travelling wave solutions of the Toda lattices via the exp-function method. As some special examples, these new exact travelling wave solutions can degenerate into the kink-type solitary wave solutions reported in [2, 16]. So using the proposed exp-function method one can obtain easily travelling wave solutions and kink-type solitary wave solutions of nonlinear differential-difference equations. Of course, lacking theory and experiments related to these solutions of the Toda lattices, we could not further say something about the real physical meaning of our exact solutions. Although these solutions are only a small part of the large variety of possible solutions for the equations discussed here, they might serve as seeding solutions for a class of localized structures which exist in these systems. We hope that they will be useful in further perturbative and numerical analysis of various solutions these lattice equations.

The presented method is only an initial work, more work will be done. Additional applications of this method to other nonlinear differential-difference systems deserve further investigation.

[1] M. J. Ablowitz, Stud. Appl. Math. **58**, 17 (1978).
[2] D. Baldwin, Ü. Göktas, and W. Hereman, Comput. Phys. Commun. **162**, 203 (2004).
[3] C. Q. Dai, Q. Yang, and J. F. Zhang, Z. Naturforsch. **59a**, 635 (2004).
[4] C. Q. Dai and J. F. Zhang, Opt. Commun. **263**, 309 (2006).
[5] J. M. Zhu, Chin. Phys. **14**, 1290 (2005).
[6] Z. Y. Yan, Non. Anal. **64**, 1798 (2006).
[7] Z. Wang and H. Q. Zhang, Chin. Phys. **15**, 2210 (2006).
[8] C. Q. Dai and J. F. Zhang, Chaos, Solitons and Fractals **27**, 1042 (2006).
[9] C. Q. Dai, J. P. Meng, and J. F. Zhang, Commun. Theor. Phys. **43**, 471 (2005).
[10] C. Q. Dai and J. F. Zhang, Int. J. Mod. Phys. B **19**, 2129 (2005).
[11] A. Khare, S. V. Dmitriev and A. Saxena, J. Phys. A: Math. Theor. **40**, 11301 (2007).
[12] J. H. He and X. H. Wu, Chaos, Solitons and Fractals **30**, 700 (2006).
[13] J. H. He and M. A. Abdou, Chaos, Solitons and Fractals **34**, 1421 (2007).
[14] W. J. Wu, Algorithms and Computation, Springer-Verlag, Berlin 1994.
[15] M. Toda, Theory of Nonlinear Lattices, Springer-Verlag, Berlin 1981.
[16] C. Q. Dai and Y. Z. Ni, Int. Theor. Phys. (2007), DOI: 10.1007/s10773-006-9285-y.
[17] Y. B. Suris, J. Phys. A: Math. Gen. **30**, 1745 (1997).